Characterization of sum automorphisms and Jordan triple automorphisms of quantum probabilistic maps

2010 J. Phys. A: Math. Theor. 43 165210

(http://iopscience.iop.org/1751-8121/43/16/165210)

IOP Publishing | Journal of Physics A: Mathematical and Theoretical

J. Phys. A: Math. Theor. **43** (2010) 165210 (9pp) doi:10.1088/1751-8113/43/16/165210

Characterization of sum automorphisms and Jordan triple automorphisms of quantum probabilistic maps*

Zhaofang Bai and Shuanping Du[1]

School of Mathematical Sciences, Xiamen University, Xiamen 361000, People's Republic of China

E-mail: shuanpingdu@yahoo.com

Received 13 October 2009, in final form 12 March 2010
Published 1 April 2010
Online at stacks.iop.org/JPhysA/43/165210

Abstract

In quantum mechanics, it is often important for the representation of a quantum system to study the structure-preserving bijective maps of the system. Such maps are also called isomorphisms or automorphisms. In this paper, using the Uhlhorn-type of Wigner's theorem, we show that both sum automorphisms and Jordan triple automorphisms of the unit ball of density operators are implemented by either unitary or anti-unitary operators of the underlying Hilbert space.

PACS numbers: 11.30.−j, 03.65.−w, 02.10.−v
Mathematics Subject Classification: 81Q10, 47N50

1. Introduction

In quantum physics, of particular importance for the representation of a physical system are structure-preserving bijective maps of the system. Such maps are also called isomorphisms or automorphisms. Automorphisms or isomorphisms are frequently amenable to mathematical formulation and can be exploited to simplify many problems. To date, they have been extensively studied in different quantum systems, and systematic theories have been achieved [9]. Recently, the most in-depth results in this field have been obtained by Monlar in a series of articles [12–15]. An overview of recent results can be found in [5, 16].

Let us now fix the notations and set the problem in mathematical terms. Let H be a separable Hilbert space with inner product $\langle , \rangle$. Let $\mathcal{B}_1(H)$ be the complex Banach space of the trace class operators on H, with the trace $\mathrm{tr}(T)$ and trace norm $\|T\|_1 = \mathrm{tr}(|T|)$, $|T| = \sqrt{T^*T}$, $T \in \mathcal{B}_1(H)$. The self-adjoint part of $\mathcal{B}_1(H)$ is denoted by $\mathcal{B}_{1r}(H)$ which is a real Banach space. By $\mathcal{B}_{1r}{}^+(H)$ we denote the positive cone of $\mathcal{B}_{1r}(H)$. As usual, the unit ball

* This work was partially supported by NNSF of China (10771175) and YNSF of Fujian (2008F3103).

[1] Author to whom any correspondence should be addressed.

of $\mathcal{B}_{1r}{}^{+}(H)$ is denoted by $S_1(H) = \{T \in \mathcal{B}_{1r}{}^{+}(H) : \mathrm{tr}(T) = \|T\|_1 \leqslant 1\}$, the surface of $S_1(H)$ by $V = \{T \in \mathcal{B}_{1r}{}^{+}(H) : \mathrm{tr}(T) = 1\}$. With reference to the quantum physical applications, $\mathcal{B}_{1r}(H)$ is called the state space, and the elements of $\mathcal{B}_{1r}{}^{+}(H)$ and V are called density operators and states, respectively (see [7, 9]). Naturally, $S_1(H)$ can be equipped with several algebraic operations. For example, one can define a partial addition on it. Namely if $T, S \in S_1(H)$ and $T+S \in S_1(H)$, then one can set $T \oplus S = T+S$. Furthermore, for $T, S \in S_1(H)$ and $\lambda \in [0, 1]$, $\lambda T + (1-\lambda)S \in S_1(H)$. Additionally, as for a multiplicative operation on $S_1(H)$, note that, in general, $T, S \in S_1(H)$ does not imply that $TS \in S_1(H)$. However, we all the time have $TST \in S_1(H)$ since $TST \in \mathcal{B}_{1r}{}^{+}(H)$ and $\mathrm{tr}(TST) = \|TST\|_1 \leqslant \|T\|_1\|S\|_1\|T\|_1 \leqslant 1$. This multiplication is a nonassociative operation and sometimes called the Jordan triple product, and also appears in infinite-dimensional holomorphy as well as in connection with the geometrical properties of C^*-algebras.

Because of the importance of $S_1(H)$, it is a natural problem to study the automorphisms of the mentioned structures. The aim of this paper is to contribute to these investigations.

The automorphisms of operator interval $[0, I]$ of positive bounded operators of H which are bounded by the identity I with the operation of partial addition were described in [4]. The automorphisms of $[0, I]$ with the operation of the Jordan triple product were investigated in [11]. In this paper, we characterize the sum automorphisms and Jordan triple automorphisms of $S_1(H)$. The core of the proof is to reduce the problem to using the Uhlhorn-type of Wigner's theorem (see [18]).

Now, let us give the concrete definitions of sum automorphism and Jordan triple automorphism. A bijective map $\Phi : S_1(H) \to S_1(H)$ is called a sum automorphism if

$$(1)\quad T+S \in S_1(H) \Leftrightarrow \Phi(T)+\Phi(S) \in S_1(H) \qquad \text{for all} \quad T, S \in S_1(H),$$

$$(2)\quad \Phi(T+S) = \Phi(T)+\Phi(S) \quad \text{whenever} \quad T+S \in S_1(H).$$

A bijective map $\Phi : S_1(H) \to S_1(H)$ is called a Jordan triple automorphism if

$$\Phi(TST) = \Phi(T)\Phi(S)\Phi(T) \quad \text{for all} \quad T, S \in S_1(H).$$

Here, it is worth mentioning that the sum automorphism has an intimate relationship with the so-called operation of $\mathcal{B}_1(H)$ (see [6, 8]), which is a fundamental notion in quantum theory. Recall that an operation Φ is a completely positive linear mapping on $\mathcal{B}_1(H)$ such that $0 \leqslant \mathrm{tr}(\Phi(T)) \leqslant 1$ for every $T \in V$. An operation represents a probabilistic state transformation. Namely if Φ is applied on an input state T, then the state transformation $T \to \Phi(T)$ occurs with the probability $\mathrm{tr}(\Phi(T))$, in which case the output state is $\frac{\Phi(T)}{\mathrm{tr}(\Phi(T))}$. By the Kraus representation theorem [8], Φ is an operation if and only if there exists a countable set of bounded linear operators $\{A_k\}$ such that $\sum_k A_k^* A_k \leqslant I$ and $\Phi(T) = \sum_k A_k T A_k^*$ holds for all $T \in \mathcal{B}_1(H)$. This is very important in describing dynamics, measurements, quantum channels, quantum interactions, quantum error, correcting codes, etc [17]. Since the operation Φ is completely positive and $0 \leqslant \mathrm{tr}(\Phi(T)) \leqslant 1$ for every $T \in V$, it is evident that such Φ maps $S_1(H)$ into $S_1(H)$ and possesses conditions (1) and (2) mentioned in the definition of sum automorphism. Thus, operations on $\mathcal{B}_1(H)$ can be reduced to maps on $S_1(H)$. From theorem 1, an explicit description can be given under the bijectivity assumption.

Our main results read as follows.

Theorem 1. *$\Phi : S_1(H) \to S_1(H)$ is a sum automorphism if and only if there exists an either unitary or antiunitary operator U on H such that $\Phi(T) = UTU^*$ for all $T \in S_1(H)$.*

We remark that, in the above result, the bijectivity assumption is indispensable to obtain a nice form of Φ. To show it, some examples originating from the Kraus representation theorem will be given after the proof of theorem 1.

Theorem 2. *$\Phi : S_1(H) \to S_1(H)$ is a Jordan triple automorphism if and only if there exists an either unitary or antiunitary operator U on H such that $\Phi(T) = UTU^*$ for all $T \in S_1(H)$.*

It is worth mentioning that, as it turns out from theorems 1 and 2, the additive and multiplicative structures of $S_1(H)$ are very closely related to each other. We remark that the question when a multiplicative function is necessary additive is important in quantum mechanics and mathematics, and was discussed for associative rings (note that our multiplication is nonassociative) in the purely algebraic setting ([10], for a recent systematic account, see [1]).

2. Proof of main results

This section is devoted to the proofs of our results. Before the proof, let us recall the general structure of density operators (see for instance [2]). For $T \in \mathcal{B}^+_{1r}(H)$, there exists an orthonormal basis $\{e_n\}_{n\in\mathbb{N}}$ of H and numbers $\lambda_n > 0$ such that

$$T = \sum_{n=1}^{+\infty} \lambda_n P_n$$

or

$$Tx = \sum_{n=1}^{+\infty} \lambda_n \langle x, e_n\rangle e_n, \qquad \forall x \in H \quad \text{and} \quad 0 < \mathrm{tr}(T) = \sum_{n=1}^{+\infty} \lambda_n < +\infty,$$

where P_n is the one-dimensional projection onto the eigenspace spanned by the eigenvector e_n. Let $\mathcal{P}_1(H)$ stand for the set of all one-dimensional projections on H. With reference to the quantum physical applications, the elements of $\mathcal{P}_1(H)$ are called pure states. We begin with a characterization of extreme points of $S_1(H)$.

Lemma 1. $\mathrm{ext}(S_1(H)) = \mathcal{P}_1(H) \cup \{0\}$*, where* $\mathrm{ext}(S_1(H))$ *denotes the set of extreme points of* $S_1(H)$.

Proof. First we prove $\mathcal{P}_1(H) \cup \{0\} \subseteq \mathrm{ext}(S_1(H))$.

Suppose $P \in \mathcal{P}_1(H)$, $P = \lambda T_1 + (1-\lambda)T_2$ for some $T_1, T_2 \in S_1(H)$ and some $\lambda \in [0, 1]$. Without loss of generality, we may assume $\lambda \neq 0, 1$. Then $P \geqslant \lambda T_1$, $P \geqslant (1-\lambda)T_2$; this implies that both T_1 and T_2 are rank one operators. Thus, $T_1 = \alpha P$ for some $\alpha \in [0, 1]$. From $1 = \mathrm{tr}(P) = \lambda\,\mathrm{tr}(T_1) + (1-\lambda)tr(T_2)$ and $\mathrm{tr}(T_1) \leqslant 1, \mathrm{tr}(T_2) \leqslant 1$, we can obtain $\mathrm{tr}(T_1) = \mathrm{tr}(T_2) = 1$. So $T_1 = P$ and similarly $T_2 = P$, i.e. P is an extreme point of $S_1(H)$. Clearly, 0 is an extreme point of $S_1(H)$. Thus, $\mathcal{P}_1(H) \cup \{0\} \subseteq \mathrm{ext}(S_1(H))$.

In order to complete the proof of this lemma, we only need to show $\mathrm{ext}(S_1(H)) \subseteq \mathcal{P}_1(H)\cup\{0\}$. In the first place, we show if $T \in S_1(H)$ and $0 < \mathrm{tr}(T) < 1$, then $T \notin \mathrm{ext}(S_1(H))$. In fact, by spectral theorem of positive operators, $T = \Sigma_{n=0}^{\infty}\alpha_n P_n$, where $\{P_n\}_{n=0}^{\infty}$ is a mutually orthogonal ($P_iP_j = 0$) sequence in $\mathcal{P}_1(H), \alpha_n \in (0, 1), 0 < \Sigma_{n=0}^{\infty}\alpha_n = \mathrm{tr}(T) < 1$, with the series converging in the trace norm of $\mathcal{B}_1(H)$. Choose the sequence $\{\beta_n\}_{n=0}^{\infty} \subseteq (0, 1)$ such that $\alpha_n < \beta_n < 2\alpha_n$ and $\Sigma_{n=0}^{\infty}\beta_n < 1$. Let $\gamma_n = 2\alpha_n - \beta_n, n = 0, 1, 2\ldots$; then $0 < \gamma_n < \alpha_n$. Set $A = \Sigma_{n=0}^{\infty}\beta_n P_n$, $B = \Sigma_{n=0}^{\infty}\gamma_n P_n$; then it is easy to see that $A, B \in S_1(H)$ and $T = \frac{1}{2}A + \frac{1}{2}B$. Therefore, T is not an extreme point of $S_1(H)$. From the above, we know that $\mathrm{ext}(S_1(H)) \subseteq \{T : T \in S_1(H), \mathrm{tr}(T) = 1\}\cup\{0\}$. For any $T \in S_1(H)$, $\mathrm{tr}(T) = 1$ and T is not a one-dimensional projection, we assert that T is not an extreme point of $S_1(H)$; this implies that $\mathrm{ext}(S_1(H)) \subseteq \mathcal{P}_1(H)\cup\{0\}$. In fact, by the spectral theorem, $T = \Sigma_{n=0}^{\infty}\alpha_n P_n$ and $\Sigma_{n=0}^{\infty}\alpha_n = 1$. Then the rank of T is at least 2 and we can assume $T = \alpha_1 P_1 + \alpha_2 P_2$ (other cases can be treated similarly). Pick up $\beta_1, \beta_2 \in (0, 1)$ such that $\alpha_1 < \beta_1 < 2\alpha_1, \alpha_2 > \beta_2$ and $\beta_1 + \beta_2 = 1$. Let $\gamma_1 = 2\alpha_1 - \beta_1, \gamma_2 = 2\alpha_2 - \beta_2$; then $\gamma_1 + \gamma_2 = 1$ and $T = \frac{1}{2}(\beta_1 P_1 + \beta_2 P_2) + \frac{1}{2}(\gamma_1 P_1 + \gamma_2 P_2)$. □

Lemma 2. *For* $P, Q \in \mathcal{P}_1(H)$, $\|P - Q\|_1 = 2\sqrt{1 - \mathrm{tr}(PQ)}$.

Proof. Suppose $P = P_x$, $Q = Q_y$, where x, y are unit vectors of H. If $PQ = 0$ or $P = Q$, it is easy to see

$$\|P - Q\|_1 = 2\sqrt{1 - \mathrm{tr}(PQ)}.$$

Thus, we assume $PQ \neq 0$ and $P \neq Q$; in this case, x, y are linearly independent. Applying Schmidt's orthogonalization, we obtain two normalized orthogonal vectors $x, \frac{x - \frac{y}{\langle y,x\rangle}}{\|x - \frac{y}{\langle y,x\rangle}\|}$. Let $\left[x, \frac{x - \frac{y}{\langle y,x\rangle}}{\|x - \frac{y}{\langle y,x\rangle}\|}\right]$ denote the linear space spanned by x and $\frac{x - \frac{y}{\langle y,x\rangle}}{\|x - \frac{y}{\langle y,x\rangle}\|}$. Note that $\|P - Q\|_1 = \mathrm{tr}(P + Q - PQ - QP)^{\frac{1}{2}}$ and let $A = P + Q - PQ - QP$; then according to the space decomposition $H = \left[x, \frac{x - \frac{y}{\langle y,x\rangle}}{\|x - \frac{y}{\langle y,x\rangle}\|}\right] \oplus \left[x, \frac{x - \frac{y}{\langle y,x\rangle}}{\|x - \frac{y}{\langle y,x\rangle}\|}\right]^{\perp}$, $A = \begin{pmatrix} * & 0 \\ 0 & 0 \end{pmatrix}$. On the other hand,

$$\begin{aligned} Ax &= (P + Q - PQ - QP)x \\ &= x + \langle x, y\rangle y - |\langle x, y\rangle|^2 x - \langle x, y\rangle y \\ &= (1 - |\langle x, y\rangle|^2)x = (1 - \mathrm{tr}(PQ))x, \end{aligned}$$

$$\begin{aligned} A\left(\frac{x - \frac{y}{\langle y,x\rangle}}{\left\|x - \frac{y}{\langle y,x\rangle}\right\|}\right) &= \frac{1}{\left\|x - \frac{y}{\langle y,x\rangle}\right\|}\left(Ax - \frac{1}{\langle y, x\rangle}Ay\right) \\ &= \frac{1}{\left\|x - \frac{y}{\langle y,x\rangle}\right\|}([1 - \mathrm{tr}(PQ)]x) - \frac{(1 - \mathrm{tr}(PQ))}{\langle y, x\rangle}y \\ &= (1 - \mathrm{tr}(PQ))\left(\frac{x - \frac{y}{\langle y,x\rangle}}{\left\|x - \frac{y}{\langle y,x\rangle}\right\|}\right). \end{aligned}$$

Therefore,

$$A = \begin{pmatrix} 1 - \mathrm{tr}(PQ) & 0 & 0 \\ 0 & 1 - \mathrm{tr}(PQ) & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

according to the space decomposition $H = [x] \oplus \left[\frac{x - \frac{y}{\langle y,x\rangle}}{\|x - \frac{y}{\langle y,x\rangle}\|}\right] \oplus \left[x, \frac{x - \frac{y}{\langle y,x\rangle}}{\|x - \frac{y}{\langle y,x\rangle}\|}\right]^{\perp}$. This implies that $\|P - Q\|_1 = \mathrm{tr}(A^{\frac{1}{2}}) = 2\sqrt{1 - \mathrm{tr}(PQ)}$, as desired.

Now, we are in a position to prove our first theorem. □

Proof of theorem 1. The sufficiency is evident, so we only need to check the necessity. We will finish the proof by checking several claims.

Claim 1. $\Phi(\lambda T + (1 - \lambda)S) = \lambda\Phi(T) + (1 - \lambda)\Phi(S)$ for all $T, S \in S_1(H)$ and $\lambda \in [0, 1]$.

We first show that $\Phi(\lambda T) = \lambda\Phi(T)$ for every $\lambda \in [0, 1]$, $T \in S_1(H)$. Clearly, $\Phi(0) = 0$. For any positive integer $p \in \mathbb{N}$, we have

$$\Phi(T) = \Phi\left(\frac{1}{p}T + \cdots + \frac{1}{p}T\right) = p\Phi\left(\frac{1}{p}T\right) \qquad (p \text{ summands}).$$

Hence, $\Phi\left(\frac{1}{p}T\right) = \frac{1}{p}\Phi(T)$. If $q \in \mathbb{N}$ with $q \leqslant p$, then

$$\Phi\left(\frac{1}{p}T + \cdots + \frac{1}{p}T\right) = q\Phi\left(\frac{1}{p}T\right) = \frac{q}{p}\Phi(T) \qquad (q \text{ summands}).$$

We can obtain that $\Phi(rT) = r\Phi(T)$ for every $r \in \mathbb{Q} \cap [0, 1]$. For $T, S \in S_1(H)$ with $S \leqslant T$, there exists $R \in S_1(H)$ such that $T = S + R$. By the additive property of Φ, we have $\Phi(T) = \Phi(S) + \Phi(R)$, and so $\Phi(S) \leqslant \Phi(T)$. Since Φ^{-1} has the same properties as Φ, we conclude that for $T, S \in S_1(H)$, $S \leqslant T \Leftrightarrow \Phi(S) \leqslant \Phi(T)$; that is, Φ preserves the order in both directions. Assume $\lambda \in [0, 1]$, $T \in S_1(H)$; then for any $x \in H$, we have

$$\begin{aligned}\langle \Phi(\lambda T)x, x\rangle &\geqslant \sup\{\langle \Phi(rT)x, x\rangle : r \leqslant \lambda, r \in \mathbb{Q} \cap [0, 1]\}\\ &= \sup\{\langle r\Phi(T)x, x\rangle : r \leqslant \lambda, r \in \mathbb{Q} \cap [0, 1]\}\\ &= \langle \Phi(T)x, x\rangle \sup\{r, r \leqslant \lambda, r \in \mathbb{Q} \cap [0, 1]\}\\ &= \langle \lambda\Phi(T)x, x\rangle,\end{aligned}$$

so $\Phi(\lambda T) \geqslant \lambda\Phi(T)$.

Similarly,

$$\begin{aligned}\langle \Phi(\lambda T)x, x\rangle &\leqslant \inf\{\langle \Phi(rT)x, x\rangle : r \geqslant \lambda, r \in \mathbb{Q} \cap [0, 1]\}\\ &= \inf\{\langle r\Phi(T)x, x\rangle : r \geqslant \lambda, r \in \mathbb{Q} \cap [0, 1]\}\\ &= \langle \Phi(T)x, x\rangle \inf\{r, r \geqslant \lambda, r \in \mathbb{Q} \cap [0, 1]\}\\ &= \langle \lambda\Phi(T)x, x\rangle,\end{aligned}$$

so $\Phi(\lambda T) \leqslant \lambda\Phi(T)$. Hence, $\Phi(\lambda T) = \lambda\Phi(T)$. For $S, T \in S_1(H)$ and $\lambda \in [0, 1]$, $\lambda T + (1 - \lambda)S \in S_1(H)$,

$$\Phi(\lambda T + (1 - \lambda)S) = \Phi(\lambda T) + \Phi((1 - \lambda)S) = \lambda\Phi(T) + (1 - \lambda)\Phi(S).$$

Claim 2. Φ is the restriction of a unique positive linear bijective operator on $\mathcal{B}_{1r}(H)$.

In the following, we will prove that Φ has a unique positive linear bijective extension from $S_1(H)$ to $\mathcal{B}_{1r}(H)$. By the proof of claim 1, for each $\lambda \in [0, 1]$ and $T \in S_1(H)$, $\Phi(\lambda T) = \lambda\Phi(T)$. For $T \in \mathcal{B}_{1r}^{+}(H)$, a natural extension of Φ from $S_1(H)$ to $\mathcal{B}_{1r}^{+}(H)$ is to define

$$\widetilde{\Phi}(T) = \|T\|_1\Phi\left(\frac{T}{\|T\|_1}\right).$$

Then for any $\lambda \geqslant 0$, one gets $\widetilde{\Phi}(\lambda T) = \lambda\widetilde{\Phi}(T)$, which is the positive homogeneity. For $T, S \in \mathcal{B}_{1r}^{+}(H)$, suppose $\widetilde{\Phi}(T) = \widetilde{\Phi}(S)$, without loss of generality, assume further $\|T\|_1 \leqslant \|S\|_1$; then $\widetilde{\Phi}\left(\frac{T}{\|S\|_1}\right) = \widetilde{\Phi}\left(\frac{S}{\|S\|_1}\right)$. Note that $\frac{T}{\|S\|_1}, \frac{S}{\|S\|_1} \in S_1(H)$, by the injectivity of Φ, $T = S$ and thus $\widetilde{\Phi}$ is injective. Since $\widetilde{\Phi}$ is positive homogeneity, the surjectivity of Φ implies that $\widetilde{\Phi}$ is surjective. So $\widetilde{\Phi} : \mathcal{B}_{1r}^{+}(H) \to \mathcal{B}_{1r}^{+}(H)$ is a bijection.

For $T_1, T_2 \in \mathcal{B}_{1r}^{+}(H)$, we can rewrite $T_1 + T_2$ in the form

$$T_1 + T_2 = (\|T_1\|_1 + \|T_2\|_1)\left(\frac{\|T_1\|_1}{\|T_1\|_1 + \|T_2\|_1}\frac{T_1}{\|T_1\|_1} + \frac{\|T_2\|_1}{\|T_1\|_1 + \|T_2\|_1}\frac{T_2}{\|T_2\|_1}\right).$$

The positive homogeneity of $\widetilde{\Phi}$ and claim 1 yield the additivity of $\widetilde{\Phi}$; that is, $\widetilde{\Phi}(T_1 + T_2) = \widetilde{\Phi}(T_1) + \widetilde{\Phi}(T_2)$.

Next, for $T \in \mathcal{B}_{1r}(H)$, write $T = T^{+} - T^{-}$, where $T^{+} = \frac{1}{2}(|T| + T), T^{-} = \frac{1}{2}(|T| - T), |T| = (T^{*}T)^{\frac{1}{2}}$. Let

$$\widehat{\Phi}(T) = \widetilde{\Phi}(T^{+}) - \widetilde{\Phi}(T^{-}).$$

Also, if $T = T_1 - T_2$ for some other $T_1, T_2 \in \mathcal{B}_{1r}^+(H)$, then $T^+ + T_2 = T^- + T_1$, by the additivity of $\widetilde{\Phi}$, $\widetilde{\Phi}(T^+) - \widetilde{\Phi}(T^-) = \widetilde{\Phi}(T_1) - \widetilde{\Phi}(T_2)$, which shows that $\widehat{\Phi}$ is well defined. Furthermore, for $T \in \mathcal{B}_{1r}(H)$, it is easy to show that $\widehat{\Phi}(-T) = -\widehat{\Phi}(T)$; combining the homogeneity of $\widetilde{\Phi}$ over the non-negative real number, we know $\widehat{\Phi}$ is linear. Assume $\widehat{\Phi}(T) = 0$, from the definition of $\widehat{\Phi}$, $\widehat{\Phi}(T^+) = \widehat{\Phi}(T^-)$, i.e. $\widetilde{\Phi}(T^+) = \widetilde{\Phi}(T^-)$. Now, the injectivity of $\widetilde{\Phi}$ implies $T^+ = T^-$, so $T = 0$ and $\widehat{\Phi}$ is injective. From the surjectivity of $\widetilde{\Phi}$ and the linearity of $\widehat{\Phi}$, it is easy to see that $\widehat{\Phi}$ is also surjective. Thus, $\widehat{\Phi}$ is a bijection on $\mathcal{B}_{1r}(H)$. In addition, a direct computation shows that $\widehat{\Phi^{-1}} = \widehat{\Phi}^{-1}$.

If $\Psi : \mathcal{B}_{1r}(H) \to \mathcal{B}_{1r}(H)$ is another positive linear map which extends Φ, then for any $T \in \mathcal{B}_{1r}(H)$,

$$\begin{aligned}\Psi(T) &= \Psi(T^+ - T^-) = \Psi(T^+) - \Psi(T^-)\\ &= \|T^+\|_1 \Psi\left(\frac{T^+}{\|T^+\|_1}\right) - \|T^-\|_1 \Psi\left(\frac{T^-}{\|T^-\|_1}\right)\\ &= \|T^+\|_1 \Phi\left(\frac{T^+}{\|T^+\|_1}\right) - \|T^-\|_1 \Phi\left(\frac{T^-}{\|T^-\|_1}\right)\\ &= \widehat{\Phi}(T^+) - \widehat{\Phi}(T^-) = \widehat{\Phi}(T).\end{aligned}$$

This shows that the extension is unique, as desired.

Claim 3. There exists an either unitary or antiunitary operator U on H such that $\Phi(T) = UTU^*$ for all $T \in S_1(H)$.

Now, $\widehat{\Phi} : \mathcal{B}_{1r}(H) \to \mathcal{B}_{1r}(H)$ is a linear bijection and preserving positive trace class operators in both directions. We assert that $\widehat{\Phi}$ is continuous in the trace norm $\|.\|_1$. For any $T \in S_1(H)$, clearly $\|\widehat{\Phi}(T)\|_1 = \|\Phi(T)\|_1 \leqslant 1$. For arbitrary $T \in \mathcal{B}_{1r}(H), \|T\|_1 \leqslant 1$, it is easy to see $T^+ \in S_1(H), T^- \in S_1(H)$. Thus $\|\widehat{\Phi}(T)\|_1 = \|\widehat{\Phi}(T^+) - \widehat{\Phi}(T^-)\|_1 \leqslant \|\widehat{\Phi}(T^+)\|_1 + \|\widehat{\Phi}(T^-)\|_1 \leqslant 2$. It follows that $\widehat{\Phi}$ is bounded on the unit ball of $\mathcal{B}_{1r}(H)$; hence, $\widehat{\Phi}$ is continuous.

In the following, we will prove that $\widehat{\Phi}$ is trace norm preserving. Firstly, it will be shown that $\widehat{\Phi}$ is trace preserving, i.e. $\mathrm{tr}(T) = \mathrm{tr}(\widehat{\Phi}(T))$ for every $T \in \mathcal{B}_{1r}(H)$. Assume $T \in \mathcal{B}_{1r}^+(H), T = \lambda_1 P_1 + \lambda_2 P_2 + \cdots + \lambda_n P_n$, where $\{P_i\}_{i=1}^n$ are mutually orthogonal one dimensional projections, $\lambda_i > 0, i = 1, 2 \ldots, n$. Then

$$\begin{aligned}\widehat{\Phi}(T) &= \lambda_1 \widehat{\Phi}(P_1) + \lambda_2 \widehat{\Phi}(P_2) + \cdots + \lambda_n \widehat{\Phi}(P_n)\\ &= \lambda_1 \Phi(P_1) + \lambda_2 \Phi(P_2) + \cdots + \lambda_n \Phi(P_n).\end{aligned}$$

By claim 1, Φ preserves the extreme point of $S_1(H)$. From lemma 1 and $\Phi(0) = 0$, we have $\Phi(\mathcal{P}_1(H)) \subseteq \mathcal{P}_1(H)$. Since Φ^{-1} has the same properties as Φ, $\mathcal{P}_1(H) \subseteq \Phi(\mathcal{P}_1(H))$, so $\Phi(\mathcal{P}_1(H)) = \mathcal{P}_1(H)$. Therefore, we can obtain $\mathrm{tr}(T) = \mathrm{tr}(\widehat{\Phi}(T)) = \Sigma_{i=1}^n \lambda_i$. For any $T \in \mathcal{B}_{1r}^+(H)$, by the spectral theorem of positive operators, there exists monotone increasing sequence $\left\{T_n = \Sigma_{i=1}^n \lambda_i P_i\right\}_{n=1}^\infty$ such that $\|T_n - T\|_1 = \mathrm{tr}(T - T_n) = \mathrm{tr}(T) - \mathrm{tr}(T_n) \to 0 (n \to \infty)$, where $\{P_i\}_{i=1}^n$ are mutually orthogonal one dimensional projections, $\lambda_i > 0, i = 1, 2, \ldots, n$. Since $\widehat{\Phi}$ is positive preserving and continuous, $\{\widehat{\Phi}(T_n)\}_{n=1}^\infty$ is monotone increasing and $\|\widehat{\Phi}(T_n) - \widehat{\Phi}(T)\|_1 = \mathrm{tr}(\widehat{\Phi}(T)) - \mathrm{tr}(\widehat{\Phi}(T_n)) \to 0 (n \to \infty)$. Note that $\mathrm{tr}(\widehat{\Phi}(T_n)) = \mathrm{tr}(T_n)$, so for every $T \in \mathcal{B}_{1r}^+(H)$, $\mathrm{tr}(T) = \mathrm{tr}(\widehat{\Phi}(T))$. For any $T \in \mathcal{B}_{1r}(H)$,

$$\mathrm{tr}(\widehat{\Phi}(T)) = \mathrm{tr}(\widehat{\Phi}(T^+)) - \mathrm{tr}(\widehat{\Phi}(T^-)) = \mathrm{tr}(T^+) - \mathrm{tr}(T^-) = \mathrm{tr}(T),$$

So $\widehat{\Phi} : \mathcal{B}_{1r}(H) \to \mathcal{B}_{1r}(H)$ is positive and trace preserving.

Next, we will show that $\widehat{\Phi}$ preserves the trace norm. In fact, for any $T \in \mathcal{B}_{1r}(H)$, we have

$$\begin{aligned}\|\widehat{\Phi}(T)\|_1 &= \|\widehat{\Phi}(T^+ - T^-)\|_1 = \|\widehat{\Phi}(T^+) - \widehat{\Phi}(T^-)\|_1 \\ &\leqslant \|\widehat{\Phi}(T^+)\|_1 + \|\widehat{\Phi}(T^-)\|_1 = \mathrm{tr}(\widehat{\Phi}(T^+)) + \mathrm{tr}(\widehat{\Phi}(T^-)) \\ &= \mathrm{tr}(T^+) + \mathrm{tr}(T^-) = \mathrm{tr}(T^+ + T^-) = \mathrm{tr}(|T|) = \|T\|_1.\end{aligned}$$

So $\widehat{\Phi} : \mathcal{B}_{1r}(H) \to \mathcal{B}_{1r}(H)$ is contractive, i.e., for $T \in \mathcal{B}_{1r}(H)$, $\|\widehat{\Phi}(T)\|_1 \leqslant \|T\|_1$. Since $\widehat{\Phi}^{-1}$ has the same properties as $\widehat{\Phi}$, we have $\|\widehat{\Phi}(T)\|_1 \geqslant \|T\|_1$ and thus $\|\widehat{\Phi}(T)\|_1 = \|T\|_1$, that is $\widehat{\Phi}$ is a $\|\cdot\|_1$-isometry of $\mathcal{B}_{1r}(H)$.

Note that, by lemma 2, for $P, Q \in \mathcal{P}_1(H)$, $PQ = 0 \Leftrightarrow \|P - Q\|_1 = 2$. Since $\widehat{\Phi}$ is trace norm preserving, we have $PQ = 0 \Leftrightarrow \widehat{\Phi}(P)\widehat{\Phi}(Q) = 0$. Now, $\widehat{\Phi}|_{\mathcal{P}_1(H)} : \mathcal{P}_1(H) \to \mathcal{P}_1(H)$ is a bijection with the property $PQ = 0 \Leftrightarrow \widehat{\Phi}(P)\widehat{\Phi}(Q) = 0$, $P, Q \in \mathcal{P}_1(H)$. Using the well-known Uhlhorn-type of Wigner's theorem (see [18]), we have $\widehat{\Phi}(P) = UPU^*(P \in \mathcal{P}_1(H))$ with some unitary or antiunitary operator on H. By the spectral theorem of self-adjoint operators and the continuity of $\widehat{\Phi}$, for all $T \in \mathcal{B}_{1r}(H)$, $\widehat{\Phi}(T) = UTU^*$; therefore, $\Phi(T) = UTU^*$ for all $T \in S_1(H)$, as desired.

Remark 1. Now, in order to illustrate that the bijective assumption is indispensable in theorem 1, we give examples which come from the Kraus representation theorem (see [3]): suppose that A_k is a finite set of bounded linear operators on H such that $\sum_k A_k A_k^* = I$, and let $\Phi(T) = \sum_k A_k^* T A_k$ $\forall T \in \mathcal{B}_{1r}^+(H)$. For $T \in \mathcal{B}_{1r}^+(H)$, there exists an orthonormal basis $\{f_j\}_{j\in\mathbb{N}}$ of H such that

$$Tx = \sum_{j=1}^{+\infty} \lambda_j \langle x, f_j\rangle f_j \qquad \forall x \in H \quad \text{and} \quad 0 \leqslant \sum_{j=1}^{+\infty} \lambda_j < +\infty.$$

Using the definition of Φ, for any orthonormal basis $\{e_i\}_{i\in\mathbb{N}}$ of H, we have

$$\begin{aligned}\sum_i \langle e_i, \Phi(T)e_i\rangle &= \sum_j \sum_i \sum_k \lambda_j \langle A_k e_i, f_j\rangle\langle f_j, A_k e_i\rangle \\ &= \sum_j \lambda_j \sum_i \sum_k \langle e_i, A_k^* f_j\rangle\langle A_k^* f_j, e_i\rangle \\ &= \sum_j \lambda_j \sum_k \langle A_k^* f_j, A_k^* f_j\rangle = \sum_j \lambda_j.\end{aligned}$$

This implies that $\mathrm{tr}(T) = \mathrm{tr}(\Phi(T))$ and so Φ is indeed a mapping which maps $S_1(H)$ into $S_1(H)$. Furthermore, it is easy to see that Φ satisfies condition (1) and condition (2) in the definition of sum automorphism. But, in general, such Φ is not a bijection and does not have a nice form as theorem 1. Now, we give a concrete example. Let $\{e_i\}_{i=1}^{\infty}$ be the orthonormal basis of H; then H can be presented as a direct sum of mutually orthogonal closed subspaces $H = \bigoplus_{k=1}^{3} H_k$ with $\dim H_k = \dim H$. Let $U_k : H \to H_k$ be unitary operators, $\omega_k \in (0, 1)$ with $\sum_{k=1}^{3} \omega_k = 1$. Set $A_k = \omega_k^{\frac{1}{2}} U_k^*$; then $\sum_{k=1}^{3} A_k A_k^* = I$ and $\Phi(T) = \sum_{k=1}^{3} A_k^* T A_k$ defines a trace preserving map of $S_1(H)$.

In the following, we will prove theorem 2.

Proof of theorem 2. The sufficiency is evident, so we only need to check the necessity. First, we show that $\Phi(\mathcal{P}_1(H)) = \mathcal{P}_1(H)$. In fact, by the spectral mapping theorem, for any $P \in \mathcal{P}_1(H)$, $\sigma(\Phi(P)) \subseteq \{0, 1\}$, where $\sigma(\Phi(P))$ denotes the spectrum of $\Phi(P)$. From the bijectivity of Φ, it is easy to see $\Phi(0) = 0$, and thus $\Phi(\mathcal{P}_1(H)) \subseteq \mathcal{P}_1(H)$. Since Φ^{-1} has the same properties as Φ, $\mathcal{P}_1(H) \subseteq \Phi(\mathcal{P}_1(H))$. Thus, $\Phi(\mathcal{P}_1(H)) = \mathcal{P}_1(H)$.

Next observe that Φ preserves the orthogonality in both directions, that is, for $T, S \in S_1(H)$, $TS = 0 \Leftrightarrow \Phi(T)\Phi(S) = 0$. Indeed, if $TS = 0$, then $\Phi(0) = \Phi(TST) = \Phi(T)\Phi(S)\Phi(T) = 0$, Thus,

$$0 = \Phi(T)\Phi(S)^{\frac{1}{2}}\Phi(S)^{\frac{1}{2}}\Phi(T) = \Phi(T)\Phi(S)^{\frac{1}{2}}(\Phi(T)\Phi(S)^{\frac{1}{2}})^*$$

which gives $\Phi(T)\Phi(S)^{\frac{1}{2}} = 0$. So $\Phi(T)\Phi(S) = 0$. Since Φ^{-1} has the same properties as Φ, we get the desired. Now $\Phi : \mathcal{P}_1(H) \to \mathcal{P}_1(H)$ is a bijection and preserves orthogonality in both directions. By the Uhlhorn-type of Wigner's theorem (see [18]), there exists a unitary or antiunitary operator U on H such that $\Phi(P) = UPU^*$ for all $P \in \mathcal{P}_1(H)$. Without loss of generality, we can assume $\Phi(P) = P$ for every $P \in \mathcal{P}_1(H)$ and then we have to prove that Φ is the identity on the whole $S_1(H)$.

In the following, we will prove $\Phi(\lambda P) = \lambda P$ for every $\lambda \in [0, 1]$ and every one-dimensional projection $P \in \mathcal{P}_1(H)$. To see this, we first show that there is a multiplicative bijection $f : [0, 1] \to [0, 1]$ such that $\Phi(\lambda P) = f(\lambda)P$. In fact, we can obtain

$$\Phi(\lambda P) = \Phi(P(\lambda P)P) = \Phi(P)\Phi(\lambda P)\Phi(P) = f_P(\lambda)P$$

for some $f_P(\lambda) \in [0, 1]$ which follows from $\Phi(P) = P$. We claim that f_P is multiplicative. For any $\mu \in [0, 1]$,

$$\begin{aligned} f_P(\lambda^2\mu)P &= \Phi(\lambda^2\mu P) = \Phi((\lambda P)(\mu P)(\lambda P)) \\ &= \Phi(\lambda P)\Phi(\mu P)\Phi(\lambda P) = f_P(\lambda)^2 f_P(\mu)P. \end{aligned}$$

Choosing $\mu = 1$, we have $f_P(\lambda^2) = f_P(\lambda)^2$, and thus $f_P(\lambda^2\mu) = f_P(\lambda^2)f_P(\mu)$. Since this holds for every $\lambda, \mu \in [0, 1]$, we conclude that f_P is multiplicative. We now claim that f_P does not depend on P. If P, Q are one-dimensional projections which are not mutually orthogonal, then $PQP \neq 0$. In this case, we have

$$\begin{aligned} f_Q(\lambda^2)\Phi(PQP) &= f_Q(\lambda^2)\Phi(P)\Phi(Q)\Phi(P) = \Phi(P)\Phi(\lambda^2 Q)\Phi(P) \\ &= \Phi(P(\lambda^2 Q)P) = \Phi((\lambda P)Q(\lambda P)) \\ &= \Phi(\lambda P)\Phi(Q)\Phi(\lambda P) = f_P(\lambda^2)\Phi(PQP). \end{aligned}$$

This gives $f_P = f_Q$. If P, Q are mutually orthogonal, then there is a one-dimensional projection R such that $PR \neq 0$ and $QR \neq 0$. Thus, we have $f_P = f_R = f_Q$. So there is a multiplicative function $f : [0, 1] \to [0, 1]$ such that $\Phi(\lambda P) = f(\lambda)P$ for every $\lambda \in [0, 1]$ and every one-dimensional projection $P \in \mathcal{P}_1(H)$. By the bijectivity of Φ, it is easy to see that f is also a bijection. In the following, we will show that f is the identity function of $[0, 1]$. It is a folklore result in the theory of functional equations that multiplicative bijections of $[0, 1]$ are exactly the functions $t \to t^s$ for some fixed real number $s > 0$. So we have $\Phi(\lambda P) = \lambda^s P, \lambda \in [0, 1]$. For $T \in S_1(H)$, $\Phi(TPT) = \Phi(T)\Phi(P)\Phi(T) = \Phi(T)P\Phi(T)$ holds for each $P \in \mathcal{P}_1(H)$; this implies that $\Phi(T) = \lambda_T T$, where λ_T is a scalar depending on T. For orthogonal one-dimensional projections $P_1, P_2 \in \mathcal{P}_1(H)$, $\Phi\left(\frac{1}{2}P_1 + \frac{1}{2}P_2\right) = \lambda_{\frac{1}{2}P_1+\frac{1}{2}P_2}\left(\frac{1}{2}P_1 + \frac{1}{2}P_2\right)$ contains

$$\Phi\left(\tfrac{1}{2}P_1 + \tfrac{1}{2}P_2\right) = P_1\Phi\left(\tfrac{1}{2}P_1 + \tfrac{1}{2}P_2\right)P_1 + P_2\Phi\left(\tfrac{1}{2}P_1 + \tfrac{1}{2}P_2\right)P_2.$$

Using the properties of Φ,

$$P_1\Phi\left(\tfrac{1}{2}P_1 + \tfrac{1}{2}P_2\right)P_1 = \Phi(P_1)\,\Phi\left(\tfrac{1}{2}P_1 + \tfrac{1}{2}P_2\right)\Phi(P_1) = \Phi\left(\tfrac{1}{2}P_1\right) = \left(\tfrac{1}{2}\right)^s P_1.$$

Similarly,

$$P_2\Phi\left(\tfrac{1}{2}P_1 + \tfrac{1}{2}P_2\right)P_2 = \left(\tfrac{1}{2}\right)^s P_2.$$

Thus,

$$\Phi\left(\tfrac{1}{2}P_1 + \tfrac{1}{2}P_2\right) = \left(\tfrac{1}{2}\right)^s P_1 + \left(\tfrac{1}{2}\right)^s P_2.$$

From $\mathrm{tr}\left(\left(\tfrac{1}{2}\right)^s P_1 + \left(\tfrac{1}{2}\right)^s P_2\right) \leqslant 1$, we have $s \geqslant 1$.

Now, for every one-dimensional projection $P \in S_1(H)$ and every $\lambda \in [0, 1]$, $\Phi(\lambda P) = \lambda^s P$, $s \geqslant 1$. Because Φ^{-1} has the same properties as Φ, we obtain that there exists $t \geqslant 1$ such that $\Phi^{-1}(\lambda P) = \lambda^t P$, $t \geqslant 1$. Note that $\Phi^{-1}(\lambda^s P) = \lambda P$, $\Phi^{-1}(\lambda^s P) = \lambda^{st} P$, so $s = t = 1$, and thus $\Phi(\lambda P) = \lambda P$, as desired.

For $T \in S_1(H)$, picking an arbitrary one-dimensional projection $P = P_x$, where x is a unit vector, we compute

$$\begin{aligned} P\Phi(T)P &= \Phi(P)\Phi(T)\Phi(P) \\ &= \Phi(PTP) = \Phi(\langle Tx, x\rangle P) \\ &= \langle Tx, x\rangle P = PTP. \end{aligned}$$

Since P is arbitrary, we get $\Phi(T) = T$ for every $T \in S_1(H)$, this completes the proof of this theorem. □

References

[1] Bai Z F and Du S P 2008 Multiplicative *-Lie isomorphisms between factors *J. Math. Anal. Appl.* **346** 327–35
[2] Blanchard Ph and Brüning E 2003 *Mathematical Methods in Physics Distrubitions Hilbert Space Operators and Variational Methods* (*Progress in Mathematical Physics*) (Boston, MA: Birkhäuser)
[3] Brüning E and Petruccione F 2010 Nonlinear positive mappings for density matrices *Physica* E **46** 436–8
[4] Cassinelli G, De Vito E, Lahti P J and Levrero A 1997 Symmetry groups in quantum mechanics and the theorem of Wigner on the symmetry transformations *Rev. Math. Phys.* **9** 921–41
[5] Cassinelli G, De Vito E, Lahti P J and Levrero A 2004 *The Theory of Symmetry Actions in Quantum Mechanics* (Berlin: Springer)
[6] Heinosaari T, Reitzner D, Stano P and Ziman M 2009 Coexistence of quantum operations *J. Phys. A: Math. Theor.* **42** 365302
[7] Kadsion R V 1965 Transformations of states in operator theory and dynamics *Topology* **3** (Suppl. 2) 177–98
[8] Kraus K 1983 *States, Effects, and Operations* (Berlin: Springer)
[9] Ludwig G 1983 *Foundations of Quantum Mechanics* vol I (Berlin: Springer)
[10] Martindale W S III 1969 When are multiplicative mappings additive? *Proc. Am. Math. Soc.* **21** 695–8
[11] Molnar L 2000 On some automorphisms of the set of effects on Hilbert space *Lett. Math. Phys.* **51** 37–45
[12] Molnar L and Pales Z 2001 ⊥-order automorphisms of Hilbert space effect algebras: the two-dimensional case *J. Math. Phys.* **42** 1907–12
[13] Molnar L 2001 Order automorphisms of the set of bounded observables *J. Math. Phys.* **42** 5904–9
[14] Molnar L 2001 Characterizations of the automorphisms of Hilbert space effect algebras *Commun. Math. Phys.* **223** 437–50
[15] Molnar L 2008 Maps on states preserving the relative entropy *J. Math. Phys.* **49** 032114
[16] Molnar L 2007 *Selected Preserver Problems on Algebraic Structures of Linear Operators and on Function Spaces* (Berlin: Springer)
[17] Nielsen M and Chuang J 2002 *Quantum Computation and Quantum Information* (Cambridge: Cambridge University Press)
[18] Uhlhorn U 1963 Representation of symmetry transformations in quantum mechanics *Ark. Fysik* **23** 307–40